Optimal Probability Partial Control of Linear Inventory Systems

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Most processes have more storage levels to control than there are production rates for changing inventories, a situation known as *partial controllability*. Necessary and sufficient conditions under which all levels can be held within predetermined limits are presented. A control law maximizing the probability of satisfactory operation is developed, together with the computational steps required for its implementation. A simple numerical example is given.

Process scheduling problems arise whenever there are more inventory levels to control than there are adjustable production rates. Given a set of fluctuating supplies and demands on such a partially controllable system, a manager needs to find a feasible production schedule, if there is one. If none exists, he would like to know this so that he can request changes or take emergency action to handle inventory excesses or deficiencies.

This article proves, for the first time, necessary and sufficient conditions for seeing if a satisfactory control action exists for a given set of disturbances. Previous work (*1, 2*), which defined partial control and justified its study, only gave looser conditions which, being necessary but not sufficient, ruled out most but not all infeasible schedules. Even though it excluded no feasible policies, this weaker condition was not as useful as the new sufficient one presented here, which generates only feasible schedules.

In addition to the existence theorem, which identifies situations impossible to control, two constructions are given for generating a feasible schedule whenever there is one. The first construction, related closely to the existence theorem, is most effective computationally when the system is almost completely controllable, that is, when there are almost as many adjustable production rates as there are levels to be controlled. The second construction is more appropriate when the system is almost completely uncontrollable because of a scarcity of manipulated variables. Both methods use an efficient technique for finding a solution to linear inequalities, known in linear programming jargon as the "Phase One procedure" (*5*).

Since a partial control system governed by these constructions will hold all inventories at satisfactory levels whenever possible, it gives maximum probability of satisfactory operation. For the linear inventory systems considered here, the feedback control action is piecewise linear and consequently requires periodic digital computation. This is not a serious objection in most process inventory systems, whose production schedules are reviewed no more frequently than once a day. For simpler, fully linear feedback systems requiring no periodic computation, see (*1*) and (*2*). Such linear systems, while not optimal, often perform acceptably well in practice.

After introducing notation and describing the problem mathematically, the article proves the main existence theorem. Then an algorithm for computing control action for an almost completely controllable system is presented and illustrated with an example simple enough to be discussed geometrically. Subsequently, two lemmas more appropriate for almost completely uncontrollable systems are proven and illustrated with the same example. Finally, the theory is extended to include controllers redundant because of linear dependence, a subject not treated in earlier work.

PROBLEM DEFINITION

Consider a linear dynamical system described by the matrix differential equation $\dot{\mathbf{x}} = \mathbf{G}\mathbf{u} + \dot{\mathbf{z}}$, where $\mathbf{x}$ is an n-dimensional state vector, $\mathbf{G}$ is an $n \times m$ matrix, $\mathbf{u}$ is an m-dimensional control vector with $n > m$, and $\dot{\mathbf{z}}$ is the n vector of random disturbances. Although the term $\dot{\mathbf{z}}$ was suppressed in the system equation in the earlier papers (*1, 2*), the present form including the $\dot{\mathbf{z}}$ term is used here for added clarity. It is assumed that the columns of the matrix $\mathbf{G}$ are independent (no loss of generality here, as will be shown in a later section of this paper) and that the state variables are bounded above and below by constants. The dynamical system is defined to be operating satisfactorily only if each state variable is within given bounds.

The problem to be solved is: given the system $\dot{\mathbf{x}} = \mathbf{G}\mathbf{u} + \dot{\mathbf{z}}$ with $\mathbf{x}^- \leq \mathbf{x} \leq \mathbf{x}^+$, where the columns of $\mathbf{G}$ are linearly independent, and given that the states of the system $\mathbf{x}$ are subject to accumulated random disturbances $\mathbf{z}(t) = \int_0^t \dot{z}(\tau)d\tau$, find the control law $\mathbf{u}^*[\mathbf{x}(t)]$ which maximizes the probability of satisfactory operation at a given time T. The probability of satisfactory operation at T is $P(T) = Pr\{\mathbf{x}^- \leq \mathbf{x}(T) \leq \mathbf{x}^+\}$ and the maximum probability is $P^*(T) = \max_{\mathbf{u}[\mathbf{x}(t)]} P(T)$. The optimal control law $\mathbf{u}^*[\mathbf{x}(t)]$ is the one which yields $P^*(T)$.

ORTHOGONALIZATION

The main idea of partial control is to transform (by linear mapping) the problem in which the state variables are partially controllable; that is, they are resultants of controllable and uncontrollable phases, to an equivalent problem, in which the controllable phases are completely separated from the uncontrollable. The following procedure for constructing such a transformation was originally developed in (*1*).

The transformation sought is represented by an $n \times n$ nonsingular matrix $\begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix}$, where $'$ denotes transposition; $\mathbf{G}$ is the given $n \times m$ system matrix with columns $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_m$; and $\mathbf{J}$ is an $n \times (n - m)$ matrix with columns $\mathbf{j}_1, \mathbf{j}_2, \ldots, \mathbf{j}_{n-m}$. The matrix $\mathbf{J}$ is obtained such that each column of $\mathbf{J}$ is orthogonal to all other columns of $\mathbf{J}$ and all columns of $\mathbf{G}$; that is, $\mathbf{j}_i' \mathbf{j}_k = \delta_{ik} = \begin{cases} 1, & i = k \\ 0 & \text{otherwise} \end{cases}$ and

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$j_i' g_h = 0$ for all i and h. Now, an m vector $\mathbf{v}$ of controllable variables and an $(n - m)$ vector $\mathbf{w}$ of uncontrollable variables are defined by $\mathbf{v} = \mathbf{G}'\mathbf{x}$ and $\mathbf{w} = \mathbf{J}'\mathbf{x}$. The components of the vector $\mathbf{v}$ are completely controllable, since

$$\dot{\mathbf{v}} = \mathbf{G}'\dot{\mathbf{x}} = \mathbf{G}'(\mathbf{G}\mathbf{u} + \dot{\mathbf{z}}) = \mathbf{G}'\mathbf{G}\mathbf{u} + \mathbf{G}'\dot{\mathbf{z}}$$

and the matrix $\mathbf{G}'\mathbf{G}$ is nonsingular. On the other hand

$$\dot{\mathbf{w}} = \mathbf{J}'\dot{\mathbf{x}} = \mathbf{J}'(\mathbf{G}\,\mathbf{u} + \dot{\mathbf{z}}) = \mathbf{J}'\dot{\mathbf{z}}$$

and since $\mathbf{w}$ is not a function of the control variables $\mathbf{u}$, it is obvious that one cannot affect $\mathbf{w}$ through the manipulation of $\mathbf{u}$. The components of $\mathbf{w}$ move at random in response to the accumulated disturbance $\mathbf{z}(t)$ by the relationship, $\mathbf{w}(t) = \mathbf{J}' \int_0^t \dot{\mathbf{z}}(\tau)\,d\tau = \mathbf{J}'\mathbf{z}(t)$.

OPTIMAL PROBABILITY PARTIAL CONTROL

In the state space, the set of all points representing satisfactory operation is $\mathcal{S} = \{\mathbf{x} : \mathbf{x}^- \leqq \mathbf{x} \leqq \mathbf{x}^+\}$, which is an n-dimensional orthotope (3) (an orthotope is an n-dimensional analogue of a rectangle). The set $\mathcal{T} = \left\{ \boldsymbol{\tau} : \boldsymbol{\tau} = \begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix} \mathbf{x}, \text{ for every } x \text{ in } \mathcal{S} \right\}$ is the image in the transformed space of the set $\mathcal{S}$ under the mapping $\begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix}$ and is an n-dimensional parallelotope (3) (the analogue of a parallelogram). Now, consider the set $\mathcal{T}_w$, defined as the orthogonal projection of the set $\mathcal{T}$ onto $\mathbf{w}$ subspace. That is, $\mathcal{T}_w = \left\{ \mathbf{w} : \mathbf{w} = \sigma\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix}, \text{ for every } \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} \text{ in } \mathcal{T} \right\}$, where σ is a linear transformation of E^n onto E^{n-m}, defined as $\sigma\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \mathbf{w}$. The central result of the article is given in the following theorem.

Theorem 1

Given the state $\mathbf{x}^1$ of the system at time t_1, satisfactory operation at t_1 is possible if and only if the point represented by the vector $\mathbf{w}^1 = \mathbf{J}'\mathbf{x}^1$ is in the set $\mathcal{T}_w$.

Proof. Suppose $\mathbf{w}^1$ is in $\mathcal{T}_w$. By the definition of $\mathcal{T}_w$, there is at least one point $\begin{pmatrix} \bar{\mathbf{v}} \\ \mathbf{w}^1 \end{pmatrix}$ in the set $\mathcal{T}$ such that $\sigma\begin{pmatrix} \bar{\mathbf{v}} \\ \mathbf{w}^1 \end{pmatrix} = \mathbf{w}^1$. Choose $\bar{\mathbf{v}}$ as the set point of $\mathbf{v}$. Then, the resulting state vector $\hat{\mathbf{x}}^1 = \begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix}^{-1} \begin{pmatrix} \bar{\mathbf{v}} \\ \mathbf{w} \end{pmatrix}$ is in $\mathcal{S}$, since the matrix $\begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix}$ represents a one-one mapping and the point $\begin{pmatrix} \bar{\mathbf{v}} \\ \mathbf{w}^1 \end{pmatrix}$ is in $\mathcal{T}$. Conversely, suppose satisfactory operation is possible; that is, for the given $\mathbf{w}^1$, there is a vector $\bar{\mathbf{v}}$ such that the point $\begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix}^{-1} \begin{pmatrix} \bar{\mathbf{v}} \\ \mathbf{w}^1 \end{pmatrix}$ is in $\mathcal{S}$. This implies $\begin{pmatrix} \bar{\mathbf{v}} \\ \mathbf{w}^1 \end{pmatrix}$ is in $\mathcal{T}$, which implies $\mathbf{w}^1$ is in $\mathcal{T}_w$.

Based on the above theorem, the optimal control law is given as follows: Whenever the point $\mathbf{w}$ is in $\mathcal{T}_w$ for the given state of the system $\mathbf{x}$, find a set point $\bar{\mathbf{v}}$ such that the point $\begin{pmatrix} \bar{\mathbf{v}} \\ \mathbf{w} \end{pmatrix}$ is in $\mathcal{T}$, and construct the error signal as $\mathbf{e} = (\bar{\mathbf{v}} - \mathbf{G}'\mathbf{x})$ to be used to drive $\mathbf{v}$ to the set point $\bar{\mathbf{v}}$. As proposed in an earlier paper (2), it is reasonable to make the rate vector $\dot{\mathbf{v}}$ proportional to this error so that all elements of the control vector $\mathbf{v}$ will reach the set point $\bar{\mathbf{v}}$ at the same time. Let K be this positive proportionality constant. Then, $\dot{\mathbf{v}} = K\mathbf{e}$, where $\dot{\mathbf{v}} = \mathbf{G}'\dot{\mathbf{x}} = \mathbf{G}'\mathbf{G}\mathbf{u} + \mathbf{G}'\dot{\mathbf{z}}$ and $K\mathbf{e} = K(\bar{\mathbf{v}} - \mathbf{G}'\mathbf{x})$ so that $\mathbf{u} = (\mathbf{G}'\mathbf{G})^{-1}[K(\bar{\mathbf{v}} - \mathbf{G}'\mathbf{x}) - \mathbf{G}'\dot{\mathbf{z}}]$ is the combined feedback-feedforward control function. In practice, $\dot{z}$ may not be known, in which case setting $\dot{\mathbf{z}}$ to its expected value, namely zero, give a feedback control function depending only on the levels $\mathbf{x}$. In view of the foregoing theorem, the control law stated above achieves satisfactory operation whenever it is possible, provided that the control system can react fast enough to random disturbances to hold the error $\mathbf{e}$ close to zero.

COMPUTING THE CONTROL LAW

In most industrial inventory systems, the difference between the number of controlled levels and the number of adjustable production rates is small. Since this difference is also the dimension of the subspace of the uncontrollable variables $\mathbf{w}$, computing schemes should be based on the $\mathbf{w}$ to remain of low dimension. Such a procedure for finding an optimal piecewise linear control law based on theorem 1 will now be presented. The reader may find the numerical example in the next section and its associated figures helpful for visualizing the sets described.

Determination of the Set $\mathcal{T}_w$

Consider all the extreme points (that is, vertexes), $\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2, \ldots, \hat{\mathbf{x}}^{2^n}$, of the set $\mathcal{S}$. Let $\mathcal{W}$ be the set composed of all the points in E^{n-m}, each of which is the image of projection of some $\hat{\mathbf{x}}^i$ on w subspace:

$$\mathcal{W} = \{\mathbf{w} \in E^{n-m} : \ \mathbf{w} = \mathbf{J}'\hat{\mathbf{x}}^i, \quad \text{for} \quad i = 1, 2, \ldots, 2^n\}$$

It can be easily shown that $\mathcal{H}(\mathcal{W}) = \mathcal{T}_w$, where $\mathcal{H}(\mathcal{W})$ is the convex hull of the set $\mathcal{W}$. The convex hull $\mathcal{H}(\mathcal{W})$ of a set $\mathcal{W}$ is defined as the smallest convex set containing all of the points in $\mathcal{W}$, and a set is said to be convex if and only if for any set of points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$ in the set, the point $\mathbf{p} = y_1\mathbf{p}_1 + y_2\mathbf{p}_2 + \ldots + y_N\mathbf{p}_N$ is also in the set for all nonnegative real numbers y_i such that $\sum_{i=1}^{N} y_i = 1$ (4).

From the definitions of convex hull and convex set, it follows that only those points in $\mathcal{W}$ which are also extreme points of $\mathcal{H}(\mathcal{W})$ are essential for the construction of $\mathcal{H}(\mathcal{W})$. In order to avoid carrying over many redundant points to subsequent calculations, we shall trim the set $\mathcal{W}$ down to $\mathcal{W}_{\min}$, where $\mathcal{W}_{\min}$ is the smallest subset of $\mathcal{W}$ with the property $\mathcal{H}(\mathcal{W}_{\min}) = \mathcal{H}(\mathcal{W})$.

Consider the following general problem: given N points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$ in E^l $(N > l)$ and a point $\mathbf{p}$ in E^l, determine whether the point $\mathbf{p}$ is in the convex hull $\mathcal{H}(\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\})$. Let $\mathbf{A}$ be the $(l + 1)$ by N matrix $\begin{pmatrix} \mathbf{p}_1 \, \mathbf{p}_2 \cdots \mathbf{p}_N \\ 1 \ \ 1 \ \ldots \ 1 \end{pmatrix}$ and $\mathbf{b}$ be the $(l + 1)$-vector $\begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix}$.

From the definition of convex hull, it follows that $\mathbf{p}$ is in $\mathcal{H}(\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\})$ if and only if there is an N vector $\mathbf{y} \geqq 0$ such that the equation $\mathbf{A}\,\mathbf{y} = \mathbf{b}$ is satisfied. So, the problem is one of determining the feasibility of the system of linear equations $\mathbf{A}\,\mathbf{y} = \mathbf{b}$ with the condition $\mathbf{y} \geqq 0$. It is a well-established fact in linear programming theory (5) that the system $\mathbf{A}\,\mathbf{y} = \mathbf{b}$ is feasible if and only if there is a basic feasible solution to $\mathbf{A}\,\mathbf{y} = \mathbf{b}$. To compute a basic feasible solution, the phase I procedure (p. 101, reference 5) of linear programming can be used.

For the problem of determining the set $T_{\min}$, the following procedure is suggested:

1. Let all the distinct points in $\mathcal{W}$ be $\hat{\mathbf{w}}^1, \hat{\mathbf{w}}^2, \ldots, \hat{\mathbf{w}}^N$ for some $N \leq 2^n$, and let $\hat{\mathcal{W}}_0 = \mathcal{W}$ and $i = 1$.
2. Let $\mathcal{W}_i = \hat{\mathcal{W}}_{i-1} - \hat{\mathbf{w}}^i$.
3. Form the system of equations $\mathbf{A}\ \mathbf{y} = \mathbf{b}$ by $\mathbf{A} = \begin{pmatrix} \hat{\mathbf{w}}^j & \ldots & \hat{\mathbf{w}}^l \\ 1 & \ldots & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \hat{\mathbf{w}}^i \\ 1 \end{pmatrix}$, where all the $\hat{\mathbf{w}}^k$'s in the matrix $\mathbf{A}$ are points in $\mathcal{W}_i$, and determine the feasibility of $\mathbf{A}\ \mathbf{y} = \mathbf{b}, \mathbf{y} \geq 0$ by phase I procedure.
4. If feasible, let $\hat{\mathcal{W}}_i = \mathcal{W}_i$; otherwise, let $\hat{\mathcal{W}}_i = \mathcal{W}_i + \hat{\mathbf{w}}^i$.
5. Let $i = i + 1$, and if $i \leq N$, return to step 2 above, otherwise, stop.

These iterations terminate when $i = N + 1$ and $\hat{\mathcal{W}}_N = \mathcal{W}_{\min}$. This procedure simply examines, point by point, whether an element of $\mathcal{W}$ is an extreme point of $\mathcal{H}(\mathcal{W})$, eliminating all but the extreme points to obtain $\mathcal{W}_{\min}$. Once the set $\mathcal{W}_{\min}$ is obtained, its elements are renamed as $\hat{\mathbf{w}}^1, \hat{\mathbf{w}}^2, \ldots \hat{\mathbf{w}}^M$ for some $M < N$ and retained for future use. These calculations need to be done only once for a given dynamical system.

Possibility of Satisfactory Operation

Since the set $T_w = \mathcal{H}(\mathcal{W}_{\min})$, by theorem 1, given the state $\mathbf{x}^1$ satisfactory operation is possible if and only if $\mathbf{J}'\mathbf{x}^1 = \mathbf{w}^1$ is in $\mathcal{H}(\mathcal{W}_{\min})$; that is, if and only if the system

$$\begin{pmatrix} \hat{\mathbf{w}}^1 & \hat{\mathbf{w}}^2 & \ldots & \hat{\mathbf{w}}^M \\ 1 & 1 & \ldots & 1 \end{pmatrix} \mathbf{y} = \begin{pmatrix} \mathbf{w}^1 \\ 1 \end{pmatrix} \tag{1}$$

has a feasible solution. The phase I procedure can be used to find a feasible solution.

Choice of Set Point

Let the n vectors $\hat{\boldsymbol{\tau}}^1, \hat{\boldsymbol{\tau}}^2, \ldots, \hat{\boldsymbol{\tau}}^M$ be the extreme points of the set T with the property $\sigma\hat{\boldsymbol{\tau}}^i = \hat{\mathbf{w}}^i$, where σ is the projection mapping used in defining the set T_w. Since the mapping σ is not one-to-one, the vectors $\hat{\boldsymbol{\tau}}^i$ are not unique in general; any particular choice of $\hat{\boldsymbol{\tau}}^i$ is satisfactory for the present purposes.

Suppose satisfactory operation is possible for the given state vector $\mathbf{x}^1$. Without loss of generality, it is assumed that the first $(n - m + 1)$ elements of $\mathbf{y}^1$ are basic variables, where $\mathbf{y}^1$ is a basic feasible solution of Equations (1). The set point vector $\bar{\mathbf{v}}$ desired must have the property that $\begin{pmatrix} \bar{\mathbf{v}} \\ \mathbf{w}^1 \end{pmatrix}$ is in the set T. Consider the m vector $\bar{\mathbf{v}}^1$ consisting of the first m elements of $\boldsymbol{\tau}$, where $\boldsymbol{\tau} = \begin{pmatrix} \bar{\mathbf{v}}^1 \\ \mathbf{w} \end{pmatrix} = (\hat{\boldsymbol{\tau}}^1, \hat{\boldsymbol{\tau}}^2, \ldots, \hat{\boldsymbol{\tau}}^M)\mathbf{y}^1$. Any convex combination, that is, $\mathbf{y}^1 \geq 0$ and $(\mathbf{y}^1)'\ \mathbf{y}^1 = 1$, of points in the convex set also is in the set T, and $\bar{\mathbf{v}}^1$ is the set point vector sought.

EXAMPLE

Consider the hypothetical chemical plant shown in Figure 1. The raw material from tank 1 is fed into the reactor through a control valve, and the reaction products are separated by distillation into products A and B in equal amounts and stored in tanks 2 and 3. Both the raw material delivery and product sales are subject to random disturbances.

Let x_1, x_2, and x_3 be the tank levels, which are bounded by ± 2, ± 1, and ± 1, respectively, and let $\hat{u}$ be the flow rate through the control valve. Then, the inventory system is described by the following differential equations:

$$\dot{x}_1 = -\hat{u} + \dot{z}_1$$

$$\dot{x}_2 = \frac{1}{2}\hat{u} + \dot{z}_2$$

$$\dot{x}_3 = \frac{1}{2}\hat{u} + \dot{z}_3$$

To normalize, let

$$u = \frac{\hat{u}}{\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]^{1/2}} = \frac{2\hat{u}}{\sqrt{6}}.$$

In terms of the new variable u, the investory system is represented by

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \mathbf{G}u + \dot{\mathbf{z}} = \begin{bmatrix} -0.816 \\ 0.408 \\ 0.408 \end{bmatrix} u + \mathbf{z}$$

The columns of $\mathbf{J}$ can be obtained by inspection (in general, they can be obtained by solving a system of algebraic equations) as

$$j_1 = \begin{bmatrix} 0.576 \\ 0.576 \\ 0.576 \end{bmatrix} \quad \text{and} \quad j_2 = \begin{bmatrix} 0 \\ 0.707 \\ -0.707 \end{bmatrix}$$

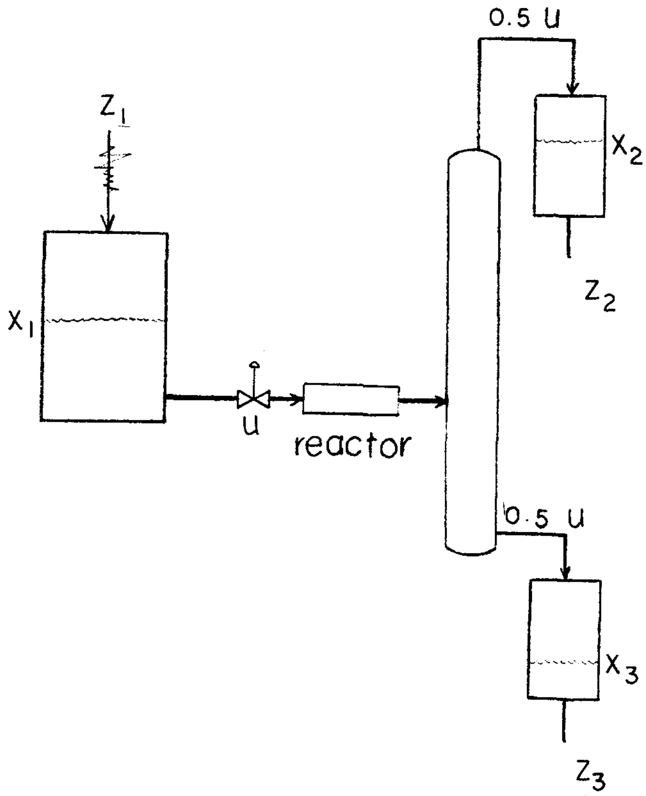

Fig. 1. System flow diagram.

In general, there are infinitely many choices for $\mathbf{J}$, given the matrix $\mathbf{G}$; any particular matrix $\mathbf{J}$ may be chosen for the purpose of partial control.

The set

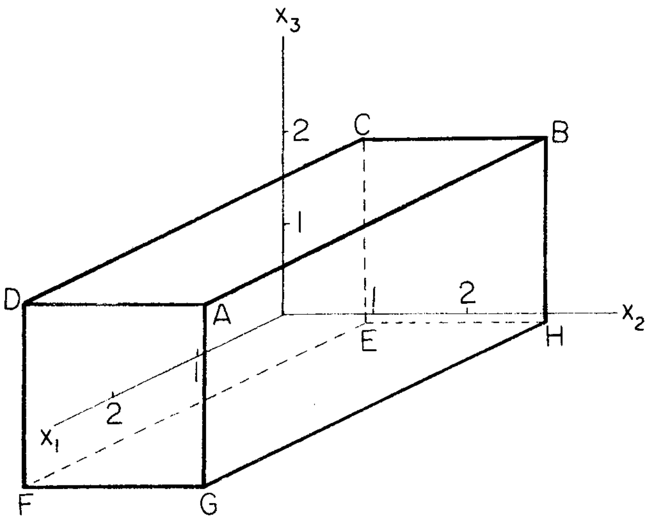

Fig. 2. Feasible region $\mathcal{S}$ in the state space.

$$\mathcal{S} = \left\{ \mathbf{x} \in E^3 : \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix} \leq \mathbf{x} \leq \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is shown in Figure 2. The set

$$T = \left\{ \begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix} \mathbf{x} : \quad \text{for all } \mathbf{x} \text{ in } \mathcal{S} \right\}$$

may be visualized by transforming a few vertexes of the set $\mathcal{S}$ such as the points A, B, C, and D in Figure 2:

$$A: \begin{bmatrix} \mathbf{G}' \\ \\ \mathbf{J}' \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.816 \\ 2.3 \\ 0 \end{bmatrix} = A';$$

$$B: \begin{bmatrix} \mathbf{G}' \\ \\ \mathbf{J}' \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.44 \\ 0 \\ 0 \end{bmatrix} = B'$$

$$C: \begin{bmatrix} \mathbf{G}' \\ \\ \mathbf{J}' \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.632 \\ -1.152 \\ -1.414 \end{bmatrix} = C';$$

$$D: \begin{bmatrix} \mathbf{G}' \\ \\ \mathbf{J}' \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.632 \\ 1.152 \\ -1.414 \end{bmatrix} = D'$$

$$\text{where } \begin{bmatrix} \mathbf{G}' \\ \\ \mathbf{J}' \end{bmatrix} = \begin{bmatrix} -0.816 & 0.408 & 0.408 \\ 0.576 & 0.576 & 0.576 \\ 0 & 0.707 & -0.707 \end{bmatrix}$$

The images of points E, F, G, and H are obtained by reflection about the origin of the points already computed above. The set T is shown in Figure 3. The set T_w is obtained by projecting the vertexes A', B', $\ldots$, and H' of T onto $\mathbf{w}$ subspace; that is, $A'' = \begin{pmatrix} 2.3 \\ 0 \end{pmatrix}$, $B'' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $C'' = \begin{pmatrix} -1.152 \\ -1.414 \end{pmatrix}$, etc. The set T_w is shown in Figure 4.

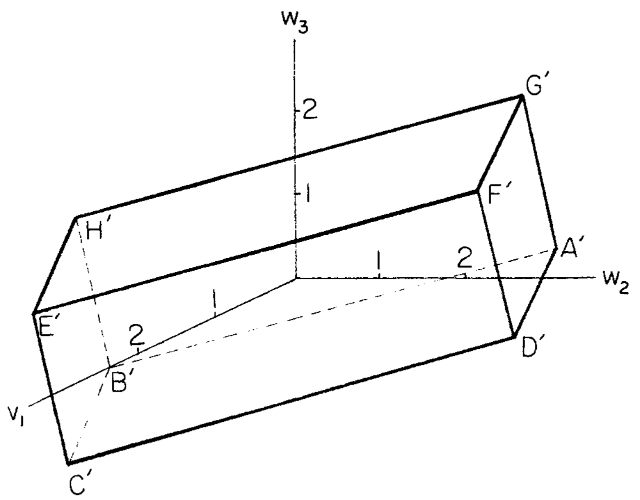

Fig. 3. Transformed region T in transformed space.

Suppose that at time t_1 the states of the system are given as $\mathbf{x}^1 = \begin{bmatrix} -1.5 \\ 0.7 \\ -1.2 \end{bmatrix}$. The values of the uncontrollable variables are calculated to be $\mathbf{w}^1 = \mathbf{J}'\mathbf{x}^1 = \begin{pmatrix} -1.152 \\ 1.34 \end{pmatrix}$. From Figure 4, it is noted that $\mathbf{w}^1$ is in the set T_w, which implies that satisfactory operation is possible. The vectors $\hat{\mathbf{w}}^1$, $\hat{\mathbf{w}}^2$, $\hat{\mathbf{w}}^3$, $\hat{\mathbf{w}}^4$, $\hat{\mathbf{w}}^5$, $\hat{\mathbf{w}}^6$ in Equations (1) are the points A'', G'', H'', E'', C'', and D'', respectively, shown in Figure 4. A basic feasible solution $\mathbf{y}^1$ of Equations (1) with $\mathbf{w}^1 = \begin{pmatrix} -1.152 \\ 1.34 \end{pmatrix}$ is $(\mathbf{y}^1)' = (0, 0.027, 0.92, 0.053, 0, 0)$. Let $\hat{\boldsymbol{\tau}}^1$, $\hat{\boldsymbol{\tau}}^2$, $\hat{\boldsymbol{\tau}}^3$, $\hat{\boldsymbol{\tau}}^4$, $\hat{\boldsymbol{\tau}}^5$, and $\hat{\boldsymbol{\tau}}^6$ be the points A', G', H', E', C', and D' respectively, as shown in Figure 3. Then, $\boldsymbol{\tau} = \begin{pmatrix} \bar{v}^1 \\ \mathbf{w} \end{pmatrix} = (\hat{\boldsymbol{\tau}}^1, \hat{\boldsymbol{\tau}}^2, \ldots, \hat{\boldsymbol{\tau}}^6)\, \mathbf{y}^1, \bar{v}^1 = 1.56$. This example will be taken up again in the next section.

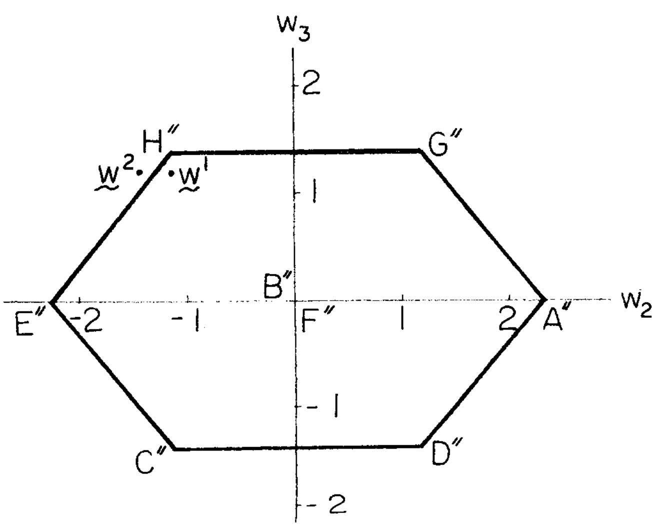

Fig. 4. Illustration of T_w.

ALTERNATE FORMS OF NECESSARY AND SUFFICIENT CONDITION

The theory presented in the previous sections was based on a geometric viewpoint (for example, the notion of the set T_w), radically different from the previous theory (*1, 2*) involving certain vector inequalities. Lemmas 1 and 2 are alternate forms of necessary and sufficient condition for satisfactory operation, which are in keeping with the previous treatments of the subject. Conditions similar to lemma 2 were proposed in past work (*1, 2*); however, the derivation in one paper (*1*) was in error, and neither work asserted the sufficiency of such conditions. The conditions derived, since they work in the v space of controllable variables, are likely to be efficient when there are few controllers available.

Consider the matrix (G, J), where G is any n by m matrix with mutually independent columns, and J is an n by $(n - m)$ matrix whose columns are mutually orthonormal and are orthogonal to all columns of G. The inverse of the nonsingular matrix $\begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix}$ is $(\overline{\mathbf{G}}, \mathbf{J})$, where $\overline{\mathbf{G}} = \mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}$, as can be readily verified. The state variables are bounded above and below by constants:

$$\mathbf{x}^- \leq \mathbf{x} \leq \mathbf{x}^+ \tag{2}$$

By definition, $\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix} \mathbf{x}$, and

$$\mathbf{x} = \begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = (\overline{\mathbf{G}}, \mathbf{J}) \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \overline{\mathbf{G}}\,\mathbf{v} + \mathbf{J}\,\mathbf{w}$$

Substitution of the above expression into inequalities (2) gives

$$\mathbf{x}^- \leq \overline{\mathbf{G}}\,\mathbf{v} + \mathbf{J}\,\mathbf{w} \leq \mathbf{x}^+$$

Given a particular $\mathbf{x}^1$ as the state of the dynamical system, the state vector can be forced to become $\mathbf{x} = \overline{\mathbf{G}}\,\mathbf{v} + \mathbf{J}\,\mathbf{J}'\mathbf{x}^1$, for any m vector v, through an appropriate use of the manipulated variables. The purpose of partial control is to use the manipulated variables in such a way, whenever possible, that the resulting state vector x is within the given bounds; that is

$$\mathbf{x}^- \leq \overline{\mathbf{G}}\,\mathbf{v} + \mathbf{J}\,\mathbf{J}'\,\mathbf{x}^1 \leq \mathbf{x}^+ \tag{3}$$

Lemma 1

Given the state $\mathbf{x}^1$ at time t_1, satisfactory operation at time t_1 is possible if and only if there is a vector v satisfying the inequalities (3).

Proof. Suppose satisfactory operation for the given $\mathbf{x}^1$ is possible; that is, there is a vector $\overline{\mathbf{x}}$ such that $\mathbf{x}^- \leq \overline{\mathbf{x}} \leq \mathbf{x}^+$ and $\mathbf{J}'\,\overline{\mathbf{x}} = \mathbf{J}'\,\mathbf{x}^1$. Let $\begin{pmatrix} \overline{\mathbf{v}} \\ \overline{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} \mathbf{G}' \\ \mathbf{J}' \end{pmatrix} \overline{\mathbf{x}}$. Then, $\overline{\mathbf{x}} = \overline{\mathbf{G}}\,\overline{\mathbf{v}} + \mathbf{J}\,\overline{\mathbf{w}}$, where $\overline{\mathbf{G}} = \mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}$. But $\mathbf{J}\,\overline{\mathbf{w}} = \mathbf{J}\,\mathbf{J}'\,\overline{\mathbf{x}} = \mathbf{J}\,\mathbf{J}'\,\mathbf{x}^1$ and $\mathbf{x}^- \leq \overline{\mathbf{G}}\,\overline{\mathbf{v}} + \mathbf{J}\,\mathbf{J}'\,\mathbf{x}^1 \leq \mathbf{x}^+$. Hence, $\overline{\mathbf{v}} = \overline{\mathbf{G}}\,\overline{\mathbf{x}}$ satisfies the inequalities (3). Conversely, suppose there is a vector $\hat{\mathbf{v}}$ satisfying the inequalities (3). Let $\hat{\mathbf{x}} = \overline{\mathbf{G}}\,\hat{\mathbf{v}} + \mathbf{J}\,\mathbf{J}'\,\mathbf{x}^1$. Then, clearly, $\mathbf{x}^- \leq \hat{\mathbf{x}} \leq \mathbf{x}^+$ and $\mathbf{J}'\,\hat{\mathbf{x}} = \mathbf{J}'\,\mathbf{x}^1$. Therefore, satisfactory operation is possible.

Next, it will be shown how one may derive from inequalities (3) a set of simple inequalities which provides yet another alternate condition for satisfactory operation. Let K be the number of m by m nonsingular submatrices $\overline{\mathbf{G}}_k$ $(k = 1, \ldots, K)$ constructable from the rows of $\overline{\mathbf{G}}$, and let $\mathbf{g}_{i,k}$ be the i^{th} row of $\overline{\mathbf{G}}_k^{-1}$ and $g_{ih,k}$ be the h^{th} element of $\mathbf{g}_{i,k}$. Let those elements of $\mathbf{x}^+$ and $\mathbf{x}^-$ corresponding to the rows of $\overline{\mathbf{G}}_k$ be assembled into m vectors $\mathbf{x}^+_k = \begin{bmatrix} x^+_{1,k} \\ x^+_{2,k} \\ \vdots \\ x^+_{m,k} \end{bmatrix}$ and $\mathbf{x}^-_k$ and define

$$X^+_{hi,k} = \begin{cases} x^+_{h,k} & \text{if} \quad \overline{g}_{ih,k} \geq 0 \\ x^-_{h,k} & \text{if} \quad \overline{g}_{ih,k} < 0 \end{cases}$$

and

$$X^-_{hi,k} = \begin{cases} x^-_{h,k} & \text{if} \quad \overline{g}_{ih,k} \geq 0 \\ x^+_{h,k} & \text{if} \quad \overline{g}_{ih,k} < 0 \end{cases}$$

Let these elements be assembled into m vectors (column) $\mathbf{X}^+_{i,k}$ and $\mathbf{X}^-_{i,k}$ and define

$$\mathbf{v}^+_k = \begin{bmatrix} \overline{\mathbf{g}}_{1,k} & \mathbf{X}^+_{1,k} \\ \overline{\mathbf{g}}_{2,k} & \mathbf{X}^+_{2,k} \\ \vdots & \vdots \\ \overline{\mathbf{g}}_{m,k} & \mathbf{X}^+_{m,k} \end{bmatrix} \quad \text{and} \quad \mathbf{v}^-_k = \begin{bmatrix} \overline{\mathbf{g}}_{1,k} & \mathbf{X}^-_{1,k} \\ \vdots & \vdots \\ \vdots & \vdots \\ \overline{\mathbf{g}}_{m,k} & \mathbf{X}^-_{m,k} \end{bmatrix}$$

For each nonsingular submatrix $\mathbf{G}_k$, consider the inequalities

$$\mathbf{v}_k^- - \overline{\mathbf{G}}_k^{-1}\,\mathbf{J}_k\mathbf{J}'\mathbf{x}^1 \leq \overline{\mathbf{v}} \leq \mathbf{v}_k^+ - \overline{\mathbf{G}}^{-1}\,\mathbf{J}_k\mathbf{J}'\mathbf{x}^1 \tag{4}$$

Define m vectors $\mathbf{V}_k^+$, $\mathbf{V}_k^-$, U, and L as

$$\mathbf{V}_k^+ = \begin{bmatrix} V^+_{1,k} \\ V^+_{2,k} \\ \vdots \\ V^+_{m,k} \end{bmatrix} = \mathbf{v}_k^+ - \overline{\mathbf{G}}_k^{-1}\mathbf{J}_k\mathbf{J}'\mathbf{x}^1;$$

$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix} = \begin{bmatrix} \min\limits_k V^+_{1,k} \\ \vdots \\ \min\limits_k V^+_{m,k} \end{bmatrix};$$

$$\mathbf{V}_k^- = \begin{bmatrix} V^-_{1,k} \\ V^-_{2,k} \\ \vdots \\ V^-_{m,k} \end{bmatrix} = \mathbf{v}_k^- - \overline{\mathbf{G}}_k^{-1}\mathbf{J}_k\mathbf{J}'\mathbf{x}^1;$$

$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{bmatrix} = \begin{bmatrix} \max\limits_k V^-_{1,k} \\ \vdots \\ \max\limits_k V^+_{m,k} \end{bmatrix}$$

Consider the sets $\mathcal{U}$ and $\overline{\mathcal{U}}$ defined as

$$\mathcal{U} = \{\mathbf{v} \in E^m : \mathbf{x}^- - \mathbf{J}\,\mathbf{J}'\mathbf{x}^1 \leq \overline{\mathbf{G}}\,\mathbf{v} \leq \mathbf{x}^+ - \mathbf{J}\,\mathbf{J}'\mathbf{x}^1\}$$

and

$$\overline{\mathcal{U}} = \{\overline{\mathbf{v}} \in E^m : \mathbf{L} \leq \overline{\mathbf{v}} \leq \mathbf{U}\}$$

Then, it is shown in the appendix that the set $\overline{\mathcal{U}}$ is an m-dimensional orthotope circumscribing the set $\mathcal{U}$. From the geometry, it is evident that the inequalities (3) have a solution; that is, $\mathcal{U}$ is nonempty, if and only if the set $\overline{\mathcal{U}}$ is nonempty. Hence, the following is true.

Lemma 2

Given a state $\mathbf{x}^1$, satisfactory operation is possible if and only if the set $\overline{\mathcal{U}} = \{\overline{\mathbf{v}} \in E^m : \mathbf{L} \leq \overline{\mathbf{v}} \leq \mathbf{U}\}$ *is* nonempty; that is, $L_i \leq U_i$ for every $i = 1, 2, \ldots, m$.

To show how the theory of this section may be applied, we now continue the discussion of the numerical example of the preceding section. Consider the same state vector $\mathbf{x}^1 = \begin{bmatrix} -1.5 \\ 0.7 \\ -1.2 \end{bmatrix}$ as before. To ascertain the possibility of satisfactory operation and to get a suitable set point, the inequalities (3) in this section may be solved. They are

$$
\begin{array}{llll}
-0.816\, \bar{v} \geqq -2 + 0.664 & & \bar{v} \leqq & 1.66 \\
0.408\, \bar{v} \geqq -1 - 0.284 & & \bar{v} \geqq & -3.14 \\
0.408\, \bar{v} \geqq -1 + 1.612 & & \bar{v} \geqq & 1.5 \\
-0.816\, \bar{v} \leqq 2 + 0.664 & & \bar{v} \geqq & -3.27 \\
0.408\, \bar{v} \leqq 1 - 0.284 & & \bar{v} \leqq & 1.75 \\
0.408\, \bar{v} \leqq 1 + 1.612 & & \bar{v} \leqq & 6.4
\end{array}
$$

One solution to the above system of inequalities is $\bar{v} = 1.58$. If $\bar{v} = 1.58$ is chosen as the set point, to which v is driven, then the resulting state $\hat{\mathbf{x}}^1 = (\mathbf{G}, \mathbf{J}) \begin{pmatrix} \bar{v} \\ \mathbf{w}^1 \end{pmatrix} = \begin{bmatrix} -1.954 \\ 0.927 \\ -0.973 \end{bmatrix}$, which represents satisfactory operation.

On the other hand, suppose that at time t_2 the state of the system is given as $\mathbf{x}^2 = \begin{bmatrix} -1.7 \\ 0.7 \\ -1.2 \end{bmatrix}$. The resulting $\mathbf{w}^2 = \mathbf{J}'\mathbf{x}^2 = \begin{pmatrix} -1.27 \\ 1.34 \end{pmatrix}$. From Figure 4, it is seen that $\mathbf{w}^2$ is not in the set T_w, which implies that satisfactory operation is not possible. Inequalities (4) for this example are

$$
\begin{aligned}
-2.45 + (0.407,\ 0.407,\ 0.407)\mathbf{x} \leqq \bar{v} \leqq 2.45 \\
+ (0.407,\ 0.407,\ 0.407)\mathbf{x} \\
-2.45 - (0.813,\ 2.306, -0.41\ )\mathbf{x} \leqq \bar{v} \leqq 2.45 \\
- (0.813,\ 2.036, -0.41\ )\mathbf{x} \\
-2.45 - (0.813, -0.41\ ,\ 2.306)\mathbf{x} \leqq \bar{v} \leqq 2.45 \\
- (0.813, -0.41\ ,\ 2.036)\mathbf{x}
\end{aligned}
$$

For $\mathbf{x}^2 = \begin{bmatrix} -1.7 \\ 0.7 \\ -1.2 \end{bmatrix}$, the above inequalities become

$$
\begin{aligned}
-3.34 \leqq \bar{v} \leqq 1.56 \\
-2.96 \leqq \bar{v} \leqq 1.93 \\
1.66 \leqq \bar{v} \leqq 6.56
\end{aligned}
$$

Since there is no $\bar{v}$ satisfying $1.66 \leqq v \leqq 1.56$, the set $\overline{\mathcal{U}}$ is empty, which also implies no satisfactory operation is possible.

CASE OF THE MATRIX G WITH DEPENDENT COLUMNS

This case was deferred until now in order to keep the main theoretical derivations simple. Suppose for the $n \times m$ system matrix $\mathbf{G}$ only k of the columns $(k < m)$ are independent. Without loss of generality, one may assume that the given matrix $\mathbf{G}$ is partitioned into $\mathbf{G} = (\mathbf{G}_i, \mathbf{G}_d)$, where the matrix $\mathbf{G}_d$ is composed of $(m - k)$ columns of $\mathbf{G}$ each of which can be expressed as linear combinations of the k columns of $\mathbf{G}_i$ as $\mathbf{G}_d = \mathbf{G}_i\mathbf{B}$, for some matrix $\mathbf{B}$. If the vector $\mathbf{u}$ of manipulated variables is partitioned into $\mathbf{u}_i$ and $\mathbf{u}_d$ likewise, then

$$
\begin{aligned}
\dot{\mathbf{x}} = \mathbf{G}\mathbf{u} + \dot{\mathbf{z}} = (\mathbf{G}_i, \mathbf{G}_d) \begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_d \end{pmatrix} + \dot{\mathbf{z}} \\
= \mathbf{G}_i\mathbf{u}_i + \mathbf{G}_d\mathbf{u}_d + \dot{\mathbf{z}} = \mathbf{G}_i\mathbf{u}_i + (\mathbf{G}_i\mathbf{B})\mathbf{u}_d + \dot{\mathbf{z}} \\
= \mathbf{G}_i(\mathbf{u}_i + \mathbf{B}\ \mathbf{u}_d) + \dot{\mathbf{z}}
\end{aligned}
$$

Define the k vector $\mathbf{u}''$ of new manipulated variables by $\mathbf{u}'' = \mathbf{u}_i + \mathbf{B}\ \mathbf{u}_d$. Then, one has $\dot{\mathbf{x}} = \mathbf{G}_i\,\mathbf{u}'' + \dot{\mathbf{z}}$, which is a problem equivalent to $\dot{\mathbf{x}} = \mathbf{G}\ \mathbf{u} + \dot{\mathbf{z}}$ with the number of manipulated variables reduced from m to k. For this new problem $\mathbf{x} = \mathbf{G}_i\mathbf{u}'' + \mathbf{z}$, the system matrix $\mathbf{G}_i$ does have independent columns and causes no problem. This analysis shows that when the matrix $\mathbf{G}$ has dependent columns, there are some superfluous manipulated variables built into the system. After a straightforward transformation of the problem, one may proceed with the general theory of partial control developed in earlier sections.

CONCLUSION

A necessary and sufficient condition on the state vector for which a linear dynamical system can be operated satisfactorily is given. Based on this condition, an optimal probability partial control law along with computational procedures necessary for implementing the control law are presented.

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NOTATION

- $\mathbf{e}$ = m vector of error signals
- h, i, k, l = indexes
- $\mathbf{g}_h$ = h^{th} column n vector of $\mathbf{G}$
- $\mathbf{j}_h$ = h^{th} column n vector of $\mathbf{J}$
- m = number of controllers
- n = number of state variables to be controlled
- $Pr(\cdot)$ = probability of
- $P(T)$ = probability of satisfactory operation at time T
- $\mathbf{u}$ = m vector of control variables
- $\mathbf{v}$ = m vector of controllable phases
- $\mathbf{w}$ = m vector of uncontrollable phases
- $\mathbf{x}$ = n vector of state variables to be controlled
- $\dot{\mathbf{z}}$ = n vector of random disturbances

Superscripts

- $+\ (-)$ = upper (lower) bound
- $*$ = optimum
- $'$ = transposition

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APPENDIX: PROOF THAT $\mathcal{V}$ CIRCUMSCRIBES $\overline{\mathcal{V}}$

Rewriting inequalities (3), we have

$$\mathbf{x}^- - \mathbf{J}\mathbf{J}'\mathbf{x}^1 \leqq \overline{\mathbf{G}}\mathbf{v} \leqq \mathbf{x}^+ - \mathbf{J}\mathbf{J}'\mathbf{x}^1 \tag{A1}$$

Consider any nonsingular $m \times m$ submatrix $\overline{\mathbf{G}}_k$ of $\mathbf{G}$ and

$$\mathbf{x}_k^- - \mathbf{J}_k\mathbf{J}'\mathbf{x}^1 \leqq \overline{\mathbf{G}}_k\mathbf{v} \leqq \mathbf{x}_k^+ - \mathbf{J}_k\mathbf{J}'\mathbf{x}^1 \tag{A2}$$

and

$$\mathbf{v}_k^- - \overline{\mathbf{G}}_k^{-1}\mathbf{J}_k\mathbf{J}'\mathbf{x}^1 \leqq \bar{\mathbf{v}} \leqq \mathbf{v}_k^+ - \overline{\mathbf{G}}_k^{-1}\mathbf{J}_k\mathbf{J}'\mathbf{x}^1 \tag{A3}$$

where all the signs are taken into account in defining v_k and v_k^+. Let the sets of all solutions v of inequalities (A2) and all solutions $\bar{v}$ of (A3) be $\mathcal{V}$ and $\bar{\mathcal{V}}$.

Proposition

The set $\mathcal{V}$ is a circumscribing interval of the set $\bar{\mathcal{V}}$.

Proof. By the definition of v_k^- and v_k^+, it is obvious that $\mathcal{V} \leq \bar{\mathcal{V}}$. Now, it suffices to show that there are solutions v of inequalities (2) of the form

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_{i-1} \\ V_i^+ \\ v_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ v_m \end{bmatrix}$$

where V_i^+ is the i^{th} element of the vector $\mathbf{V}^+ = \mathbf{v}_k^+ - \bar{\mathbf{G}}_k^{-1}\mathbf{J}_k\mathbf{J}'\mathbf{x}^1$.

Introducing nonnegative slack vectors y and z in inequalities (A2), we get

$$\bar{\mathbf{G}}_k\mathbf{v} + \mathbf{I}\mathbf{y} = \mathbf{x}_k^+ - \mathbf{J}_k\mathbf{J}'\mathbf{x}^1 \qquad \text{(A4)}$$

and

$$\bar{\mathbf{G}}_k\mathbf{v} - \mathbf{I}\mathbf{z} = \mathbf{x}_k^- - \mathbf{J}_k\mathbf{J}'\mathbf{x}^1 \qquad \text{(A5)}$$

Let the i^{th} row of $\bar{\mathbf{G}}_k^{-1}$ be $\bar{\mathbf{g}}_{i,k}$, and first assume that all the elements of the row vector $\bar{\mathbf{g}}_{i,k}$ are nonnegative. Premultiplying (A4) by $\bar{\mathbf{G}}_k^{-1}$, we get

$$\mathbf{v} + \bar{\mathbf{G}}^{-1}\mathbf{y} = \bar{\mathbf{G}}^{-1}\mathbf{x}_k^+ - \mathbf{G}_k^{-1}\mathbf{J}_k\mathbf{J}'\mathbf{x}^1$$

Let $\mathbf{y} = \mathbf{0}$ (nonbasic variables). Then

$$\mathbf{v} = \bar{\mathbf{G}}_k^{-1}\mathbf{x}_k^+ - \mathbf{G}_k^{-1}\mathbf{J}_k\mathbf{J}'\mathbf{x}^1$$

is a solution of (A4), and it is of the form desired; that is

$$\mathbf{v}' = (v_1, v_2, \ldots, v_{i-1}, V_i^+, v_{i+1}, \ldots, v_m)$$

Substitution

$$\mathbf{v} = \bar{\mathbf{G}}^{-1}(\mathbf{x}_k^+ - \mathbf{J}_k\mathbf{J}'\mathbf{x}^1)$$

into (A5) yields

$$\mathbf{x}_k^+ - \mathbf{I}\mathbf{z} = \mathbf{x}_k^-$$

and, since $\mathbf{x}_k^- \leq \mathbf{x}_k^+$, there is some nonnegative z satisfying the above equation. Then

$$\mathbf{v} = \bar{\mathbf{G}}_k^{-1}(\mathbf{x}_k^+ - \mathbf{J}_k\mathbf{J}'\mathbf{x}^1)$$

is a solution of (A4) and (A5), hence, of the inequalities (A2). Now, for the general case where the elements of the row vector $\bar{\mathbf{g}}_{i,k}$ are of mixed signs, with no loss of generality, assume

$$\bar{\mathbf{g}}_{i,k} = (-, +, +, \ldots, +)$$

Consider the following systems of equations:

$$\begin{array}{llll}
\bar{g}_{11,k}v_1 + g_{12,k}v_2 + \ldots & + \bar{g}_{1m,k}v_m - z_1 & & = x^-_{1,k} - \ldots \\
\bar{g}_{21,k}v_1 + \ldots & + \bar{g}_{2m,k}v_m + y_2 & & = x^+_{2,k} - \ldots \\
\vdots & \ddots & & \\
\bar{g}_{m1,k}v_1 + \ldots & + \bar{g}_{mm,k}v_m & + y_m & = x^+_{m,k} - \ldots
\end{array} \qquad \text{(A6)}$$

$$\begin{array}{llll}
\bar{g}_{11,k}v_1 + \ldots & + \bar{g}_{1m,k}v_m + y_1 & & = x^+_{1,k} - \ldots \\
\bar{g}_{21,k}v_1 + \ldots & + \bar{g}_{2m,k}v_m - z_2 & & = x^+_{1,k} - \ldots \\
\vdots & \ddots & & \\
\bar{g}_{m1,k}v_1 + \ldots & + \bar{g}_{mm,k}v_m & - z_m & = x^-_{m,k} - \ldots
\end{array} \qquad \text{(A7)}$$

Again, premultiplying (A6) by $\bar{\mathbf{G}}_k^{-1}$ and letting $(z_1, y_2, y_3, \ldots, y_m) = \mathbf{0}$, we get a basic feasible solution v of (A6) in the form desired, that is, $(v_1, v_2, \ldots, V_i^+, \ldots, v_m)'$, and it is easily verified to be a solution also of (A7). Hence, there exists a solution v of (A2) of the form $(v_1, v_2, \ldots, V_i^+, \ldots, v_m)'$ for any i. Therefore, $\bar{\mathcal{V}}$ is a circumscribing interval of $\mathcal{V}$.

The Purification of Helium Gas by Physical Adsorption at 76°K.

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The physical adsorption isotherms for three methane-helium mixtures, two nitrogen-helium mixtures, and one methane-nitrogen-helium mixture were measured at 76°K. and pressures of 2 to 65 atm. on a coconut shell charcoal. The adsorption isotherms of the pure components (nitrogen, methane, and helium) were also determined over the appropriate pressure ranges.

Methods for predicting the mixture adsorption isotherms by using only the pure component isotherms are discussed and are shown to be adequate for these systems.

The concentration vs. time or breakthrough curves were also measured for both the binary and ternary mixtures at a number of different flow rates. Mass transfer coefficients for both the gas phase and the adsorbed phase were obtained from these breakthrough curves.

Helium, once a laboratory curiosity, is now an important industrial gas. Helium usage was 948 million cu. ft. in 1966, and commercial consumption is expected to grow at about 15%/yr. in the future (*1*).

The facilities that produce high purity helium use a physical adsorption process at liquid nitrogen temperature for removal of the last quantities of nitrogen or hydrocarbon impurities. However, there are no published data on either the equilibrium adsorption capacities of commercial adsorbents for nitrogen or hydrocarbon impurities, or on the kinetic processes taking place in the adsorption